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THE SCHRÖDINGER OPERATOR WITH GROWING POTENTIAL[*]

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Abstract. The Schrödinger operator with a quadratically growing potential on the positive half-line with the Dirichlet boundary condition at zero is studied. An explicit form of the inverse operator is found. Using the inverse operator, a description of the considered operator is given.

Keywords: The Schrödinger operator, growing potential, the Weber function, Dirichlet boundary condition.
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1 Introduction

Quantum mechanics gave a powerful impetus to the development of the spectral theory of differential operators with increasing coefficients. In particular, various spectral problems for the Schrödinger operator with increasing potential were studied very extensively (Abbasova & Khanmamedov, 2022; Bagirova & Khanmamedov, 2018; Guseinov et al., 2018; Gafarova et al., 2021; Korotyaev, 2018; Savchuk & Shkalikov, 2017; Chelkak et al., 2004; Chelkak & Korotyaev, 2007). In some problems of conformal field theory are closely related to the Schrödinger operator with an quadratic potential (Sahnovich, 1964). However, the question of describing the domain of definition of such operators has not been studied previously.

We consider the operator L defined on the space $L_2\left(0,+\infty\right)$ by the differential expression

$$l\left(y\right)=-y''+\left(x^2+x\right)y,\, x\in\left[0,+\infty\right)$$

with the domain

$$D\left(L\right)=\left\{y\in L_2\left(0,+\infty\right):y\in W^2_{2,loc},\, l\left(y\right)\in L_2\left(0,+\infty\right),\, y\left(0\right)=0\right\}.$$

In this paper we describe the domain of definition of the operator L.

Consider the equation

$$-y''+\left(x^2+x\right)y=\lambda y. \tag{1}$$

If we set $z=x+\frac{1}{2}$, $u\ (z)=y\left(z-\frac{1}{2}\right)$, $\nu=\lambda+\frac{1}{4}$, then equation (1) takes the form

$$-u''\left(z\right)+z^2u\left(z\right)=\nu\, u\left(z\right).$$

It is well known (see (Abramowitz & Stegun, 1964)) that this equation has two linearly independent solutions $D_{\frac{\nu-1}{2}}\left(\sqrt{2}z\right)$ and $D_{\frac{\nu-1}{2}}\left(-\sqrt{2}z\right)$, where $D_\nu\left(z\right)$ is Weber function. Therefore, the

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equation (1) has two solutions $\psi(x,\lambda)=D_{\frac{4\lambda-3}{8}}\left(\sqrt{2}\left(x+\frac{1}{2}\right)\right)$ and $\Psi(x,\lambda)=D_{\frac{4\lambda-3}{8}}\left(-\sqrt{2}\left(x+\frac{1}{2}\right)\right)$. For each x both these functions are entire functions of the index λ. Moreover, the solutions $\psi(x,\lambda)=D_{\frac{4\lambda-3}{8}}\left(\sqrt{2}\left(x+\frac{1}{2}\right)\right)$ and $\Psi(x,\lambda)=D_{\frac{4\lambda-3}{8}}\left(-\sqrt{2}\left(x+\frac{1}{2}\right)\right)$ for $\lambda\neq 2n+\frac{3}{4}, n=0,1,2,...$ are linearly independent, because their Wronskian is equal (Abramowitz & Stegun, 1964) to $W(\lambda)=2\sqrt{\pi}\Gamma^{-1}\left(\frac{3-4\lambda}{8}\right)$, where $\Gamma(\cdot)$ is Gamma function. Since $x^2+x\to+\infty$ as $x\to+\infty$, it follows that the operator L has a purely discrete spectrum consisting (Sahnovich, 1964) of simple eigenvalues $\lambda_n>0$, $n=1,2,...$, where $\lambda_n\to+\infty$ for $n\to+\infty$. Moreover, the eigenvalues λ_n, $n=1,2,...$, are the roots of the function $\psi(x,\lambda)$. We need some (Abramowitz & Stegun, 1964) asymptotic equalities related to the functions and $\psi(x,\lambda)$ and $\Psi(x,\lambda)$

$$\psi(x,\lambda)=\left(\sqrt{2}\left(x+\frac{1}{2}\right)\right)^{\frac{\lambda}{2}}e^{-\frac{x^2}{2}}\left(1+O\left(x^{-2}\right)\right),\quad x\to+\infty, \tag{2}$$

$$\psi'(x,\lambda)=-\frac{1}{\sqrt{2}}\left(\sqrt{2}\left(x+\frac{1}{2}\right)\right)^{\frac{\lambda+2}{2}}e^{-\frac{x^2}{2}}\left(1+O\left(x^{-2}\right)\right),\quad x\to+\infty, \tag{3}$$

$$\Psi(x,\lambda)=\frac{\sqrt{2\pi}}{\Gamma\left(\frac{3-4\lambda}{16}\right)}\left(\sqrt{2}\left(x+\frac{1}{2}\right)\right)^{-\frac{\lambda+2}{2}}e^{\frac{x^2}{2}}\left(1+O\left(x^{-2}\right)\right),\quad x\to+\infty, \tag{4}$$

$$\Psi'(x,\lambda)=\frac{1}{\sqrt{2}}\frac{\sqrt{2\pi}}{\Gamma\left(\frac{3-4\lambda}{16}\right)}\left(\sqrt{2}\left(x+\frac{1}{2}\right)\right)^{-\frac{\lambda+2}{2}}e^{\frac{x^2}{2}}\left(1+O\left(x^{-2}\right)\right),\quad x\to+\infty. \tag{5}$$

We also introduce the special solutions

$$\psi(x)=\psi(x,0), \tag{6}$$

$$\varphi(x)=W^{-1}(0)\left[\Psi(x,0)-\frac{\Psi(0,0)}{\psi(0,0)}\psi(x,0)\right] \tag{7}$$

of the equation (1) with $\lambda=0$.

2 The main result

The main result of this paper is the following theorem.

Theorem 1. *The domain $D(L)$ coincides with the set of functions of the form*

$$y(x)=\psi(x)\int_0^x\varphi(t)f(t)\,dt+\varphi(x)\int_x^{+\infty}\psi(t)f(t)\,dt, \tag{8}$$

where $f(x)$ ranges over the entire space $L_2(0,+\infty)$. For each function $y(x)\in D(L)$, one has

$$xy(x)\to 0,\ y'(x)\to 0,\ xy(x),y'(x)\in L_2(0,+\infty). \tag{9}$$

Equation (8) defines a bounded operator on $L_2(0,+\infty)$, which is the inverse of L.

Proof. Obviously, the operator L is densely defined, because its domain contains infinitely differentiable functions compactly supported on $(0,\infty)$; the set of these functions is well known to be dense in $L_2(0,\infty)$. Moreover, the operator L is self-adjoint. Further, note that by virtue of (2), (6), the function $\psi(x)=D_{-\frac{3}{8}}\left(\sqrt{2}\left(x+\frac{1}{2}\right)\right)$ decays like exponent as $x\to\infty$. Hence the improper integral in (8) converges. Since $f(x)\in L_2(0,+\infty)$, it follows that the function $y=y(x)$ defined in (8) lies in $W_2^1[0,b]$ for every finite b. By differentiating, we obtain

$$y'(x)=\psi'(x)\int_0^x\varphi(t)f(t)\,dt+\varphi'(x)\int_x^{+\infty}\psi(t)f(t)\,dt, \tag{10}$$

whence it follows that $y'(x) \in W_2^1[0,b]$ for every finite b. By differentiating once more, we obtain

$$y''(x) = \left[\psi'(x)\varphi(x) - \psi(x)\varphi'(x)\right] f(x) + \psi''(x)\int_0^x \varphi(t) f(t)\, dt +$$

$$+\varphi''(x)\int_x^{+\infty} \psi(t) f(t)\, dt = -f(x) + \left(x^2 + x\right) y(x),$$

i.e., $y(x) \in W_2^2[0,b]$ for for each $b > 0$ and $\ell(y) = f(x) \in L_2(0,+\infty)$. Since $y(0) = 0$, it follows that $y(x) \in D(L)$. The converse is true as well. Namely, let $y \in D(L)$ and $\ell(y) = f(x) \in L_2(0,+\infty)$. By a classical theorem on the general form of a solution of a differential equation,

$$y(x) = C_1\psi(x) + C_2\varphi(x) + \psi(x)\int_0^x \varphi(t) f(t)\, dt + \varphi(x)\int_x^{+\infty} \psi(t) f(t)\, dt,$$

where C_1 and C_2 are constants. It follows from the relation $y(0) = 0$ that $C_1 = 0$, while the condition $y \in L_2(0,+\infty)$ and the estimate (1), which will be proved below, imply that $C_2 = 0$; i.e., y admits the representation (8). Thus, formula (8) defines the inverse operator L^{-1}. Its boundedness follows from the estimate $|y(x)| \leq R\|f\|$ on every finite interval $[0,b]$ and the estimate (11). Here and in what follows, the letter $R, R_j, j = 1,2,3,4,5$ stands for various positive constants, and $\|\circ\| = \|\circ\|_{L_2(0,\infty)}$.

Let us prove relations (9). First, note that, by virtue of (4), (7), there exists a constant R such that the estimate

$$|\varphi(t)| \leq Rt^{--1}e^{\frac{t^2}{2}}$$

holds for $t > 0$. It is easily seen that the function $g(t) = t^{-1}e^{\frac{t^2}{2}}$ increases for sufficiently large $t > b$. Hence

$$\left|\int_0^x \varphi(t)\ f(t)\, dt\right| \leq \left(\int_0^b + \int_b^{x-1} + \int_{x-1}^x\right) |\varphi(t)|\, |f(t)| \leq R_1\|f\| +$$

$$+Rx^{\frac{1}{2}}x^{-1}e^{\frac{(x-1)^2}{2}}\|f\| + Rx^{-1}e^{\frac{x^2}{2}}\left(\int_{x-1}^x |f(t)|^2\, dt\right)^{\frac{1}{2}}$$

for $x > b+1$. Here the constant R_1 depends on b alone. This estimate, together with the representation (2), implies that

$$\left|\psi(x)\int_0^x \varphi(t)\ f(t)\, dt\right| \leq R_2\|f\|e^{-\frac{x^2}{2}} +$$

$$+R_2x^{-\frac{1}{2}}e^{-x+\frac{1}{2}}\|f\| + R_2x^{-1}\left(\int_{x-1}^x |f(t)|^2\, dt\right)^{\frac{1}{2}}$$

for sufficiently large x.We have estimated the first summand in (8). In a similar way, we estimate the second summand. The function $h(t) = te^{-\frac{t^2}{2}}$ is decreasing for sufficiently large t, and hence

$$\left|\int_x^\infty \psi(t)\ f(t)\, dt\right| \leq \left(\int_x^{x+1} + \int_{x+1}^\infty\right) |\psi(t)|\, |f(t)| \leq R\int_x^{x+1} h(t)\, t^{-1}|f(t)|\, dt +$$

$$+R\left(\int_{x+1}^\infty h^2(t)\, t^{-2}dt\right)^{\frac{1}{2}}\|f\| \leq Rh(x)\, x^{-1}\left(\int_x^{x+1} |f(t)|^2\, dt\right)^{\frac{1}{2}} + Rh(x+1)\, x^{-1}\|f\|$$

for large x. We have

$$|\varphi(x)| \leq R\, h^{-1}(x),\ h(x+1)\, h^{-1}(x) = \left(1 + \frac{1}{x}\right) e^{-x-\frac{1}{2}}$$

and hence the absolute value of the second term on the right-hand side in (8) can be estimated by

$$Rx^{-1}\left(\int_x^{x+1}|f(t)|^2\,dt\right)+R\left(1+\frac{1}{x}\right)x^{-1}e^{-x-\frac{1}{2}}\,\|f\|\,.$$

By adding the resulting estimates, we arrive at the inequality

$$\begin{aligned}&|y(x)|\le R_3\,\|f\|\,e^{-\frac{x^2}{2}}+\\&+R_3x^{-\frac{1}{2}}e^{-x}\,\|f\|+R_3x^{-1}\left(\int_{x-1}^{x+1}|f(t)|^2\,dt\right)^{\frac{1}{2}},\ x>b+1,\end{aligned}\tag{11}$$

which proves the first relation in (9). The second relation in (9) can be obtained in the same way except that (11) is used instead of (8) and we take into account the fact that the estimates for the derivatives $\psi(x)$ and $\varphi(x)$ differ from the estimates for the functions themselves by the factor x.

Let us prove that $xy(x)\in L_2(0,+\infty)$. It follows from the estimate (11) that

$$\int_b^\infty|xy(x)|^2\,dx\le R_4\,\|f\|^2\int_b^\infty x^2e^{-x^2}dx\,+R_4\,\|f\|^2\int_b^\infty xe^{-2x}dx+$$

$$+R_4\int_b^\infty\int_{x-1}^{x+1}|f(t)|^2\,dtdx\le R_5\,\|f\|^2+R_5\int_{b-1}^\infty|f(t)|^2\int_{t-1}^{t+1}dxdt\le 3R_5\,\|f\|^2\,.$$

The relations $y'(x)\in L_2(0,+\infty)\,,\ y'(x)\to 0, x\to\infty$ can be obtained by virtue of (3), (5)-(7), (10) in a similar way with regard to the fact that $|y'(x)|$ is bounded by the right-hand side of (11) multiplied by x. This completes the proof of the theorem.

The results obtained can be used to study the spectral properties of the operator L. Moreover, these results can be extended to the case of the operator L_1 defined on the space $L_2(0,+\infty)$ by the differential expression

$$l_1(y)=-y''+\left(x^2+2x\right)y+q(x)\,y,\ x\in[0,+\infty)$$

with the domain

$$D(L)=\left\{y\in L_2(0,+\infty):y\in W_{2,loc}^2,\ l(y)\in L_2(0,+\infty)\,,\ y(0)=0\right\},$$

where the function $q(x)$ satisfies the condition $\int_0^{+\infty}|q(x)|\,dx<\infty$.

□

3 Conclusion

In the present paper, we have given a description of the domain of definition of the one-dimensional Schrödinger operator on the positive half-line with the Dirichlet boundary condition at zero. In fact, the potential grows quadratically. We have applied the classical method of finding the resolvent of a differential operator. We have also found an explicit form of the operator which is the inverse of the original operator. The results obtained can be used in the study of direct and inverse spectral problems for the considered operator.

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